Alternative Randomization for Valuing American Options (Mathematical Programming Concerning Decision Makings and Uncertainties)

Kimura, Toshikazu

数理解析研究所講究録 1373 巻 2004 年 98-105

Alternative Randomization for Valuing American Options*

北海道大学・経済学研究科　木村 俊一 (Toshikazu Kimura)
Graduate School of Economics
Hokkaido University

1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes [1] and Merton [8]. Although a vast majority of traded options are of American-style optimally exercised before the maturity, there are no closed-form formulas for their values even in the standard model. The principal difficulty in analyzing American options may be the absence of an explicit expression for the *early exercise boundary*, which is an optimal level of critical asset value where early exercise occurs.

Due to the lack of closed-form formulas for American option values, many approximate and/or numerical solutions have been developed so far. Broadie and Detemple [2] numerically evaluated recent methods for computing American option values. From those numerical experiments, it comes out that a numerical procedure developed by Carr [3] is fast and accurate among existing methods. Carr's procedure is based on valuing an American option with a randomized maturity, so that it is called the *randomization* approach. The purpose of this paper is to improve Carr's randomization approach by introducing alternative randomization based on an order statistic from an exponential population.

2 Free Boundary Problem

Let $(S_t)_{t\geq 0}$ be the stock price governed by the risk-neutralized diffusion process

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t, \qquad t \geq 0, \tag{2.1}$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns, and $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider an American *put* option written on $(S_t)_{t\geq 0}$, which has maturity date T and strike price K. Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \qquad 0 \leq t \leq T,$$

denote the value of the American put option at time t. See Remark 1 below for the call value.

*This paper is an abbreviated version of Kimura [5].

McKean [7] showed that the alive American put value P and an early exercise boundary $(B_t)_{t\in[0,T]}$ can be jointly obtained by solving a *free boundary problem*, which is specified by the Black-Scholes-Merton PDE

$$\tfrac{1}{2}\sigma^2 S^2 P_{SS} + (r-\delta)SP_S - rP + P_t = 0, \qquad S > B_t, \tag{2.2}$$

together with the boundary conditions

$$\lim_{S\uparrow\infty} P(t,S) = 0, \tag{2.3}$$

$$\lim_{S\downarrow B_t} P(t,S) = K - B_t, \tag{2.4}$$

$$\lim_{S\downarrow B_t} P_S(t,S) = -1, \tag{2.5}$$

and the terminal condition

$$P(T,S) = (K-S)^+. \tag{2.6}$$

The condition (2.4) is often called the *value-matching condition* and (2.5) is called the *smooth-pasting* or *high-contact condition*.

It is sometimes convenient to work with the equations where the current time t is replaced by the remaining time until maturity $s \equiv T - t$. From (2.2)–(2.6), the put price for the reversed process $\hat{P}(s, S_s) \equiv P(T-s, S_{T-s})$ satisifies the PDE

$$\tfrac{1}{2}\sigma^2 S^2 \hat{P}_{SS} + (r-\delta)S\hat{P}_S - r\hat{P} - \hat{P}_s = 0, \qquad S > \hat{B}_s, \tag{2.7}$$

with the boundary conditions

$$\lim_{S\uparrow\infty} \hat{P}(s,S) = 0, \tag{2.8}$$

$$\lim_{S\downarrow \hat{B}_s} \hat{P}(s,S) = K - \hat{B}_s, \tag{2.9}$$

$$\lim_{S\downarrow \hat{B}_s} \hat{P}_S(s,S) = -1, \tag{2.10}$$

and the initial condition

$$\hat{P}(0,S) = (K-S)^+. \tag{2.11}$$

Remark 1 Let $C(0,S;K,r,\delta)$ denote the initial value of the associated American call option with the same parameters as those in the put option. McDonald and Schroder [6] proved the *parity relation*

$$C(0,S;K,r,\delta) = P(0,K;S,\delta,r). \tag{2.12}$$

Let $B_t^P \equiv B_t^P(K,r,\delta)$ and $B_t^C \equiv B_t^C(K,r,\delta)$ denote the early exercise boundaries of the American put and call options, respectively. Carr and Chesney [4] showed *symmetry relation*

$$B_t^C(K,r,\delta) = \frac{K^2}{B_t^P(K,\delta,r)}. \tag{2.13}$$

Due to the parity/symmetry relations, the results for the American call option can be derived from the associated put option.

3 Randomization Approach

Carr [3] developed a valuing method for the American put. Carr's randomization approach consists of the following steps:

1. Randomize the maturity date by an *exponentially* distributed random variable $\widetilde{T}$ with mean $E[\widetilde{T}] = \lambda^{-1} = T$ in order to value the so-called *Canadian option*.
2. Extend the result to the case that $\widetilde{T}$ is distributed as the *n-stage Erlangian* distribution with the same mean $E[\widetilde{T}] = T$.
3. Take the limit of the randomized option value by letting $n \to \infty$ to obtain the underlying American option value.

To understand the meaning of the step 3 above, Figure 1 illustrates the convergence of the n-stage Erlangian distribution to Dirac's delta function concentrated at the mean $\lambda^{-1} = 1$.

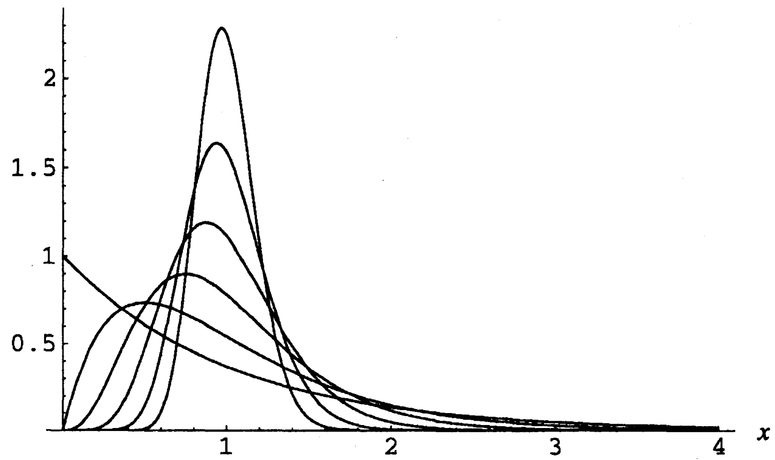


Figure 1: n-stage Erlangian pdf ($\lambda^{-1} = 1$, $n = 1, 2, 4, 8, 16, 32$)

Actually, the idea of Carr's randomization is *not* new. In the theory of integral transforms, this idea goes by the name of the Post-Widder inversion formula [9]: For a continuous function $g(t)$ $(t \geq 0)$, define

$$g_n^*(T) = \int_0^\infty g(t) \frac{(nt/T)^{n-1}}{(n-1)!} \frac{n}{T} e^{-nt/T} dt. \tag{3.1}$$

Then, we have

$$\lim_{n \to \infty} g_n^*(T) = g(T), \tag{3.2}$$

which is the essential point of Carr's randomization method.

For $\lambda > 0$, let

$$P^* \equiv P^*(\lambda, S) = \int_0^\infty \lambda e^{-\lambda s} \hat{P}(s, S) ds \tag{3.3}$$

be the Laplace-Carson transform (LCT) of $\hat{P}(s,S)$. Then, from (2.7)-(2.11), $P^*(\lambda,S)$ satisfies the ODE

$$\tfrac{1}{2}\sigma^2 S^2 P^*_{SS} + (r-\delta)SP^*_S - (\lambda+r)P^* + \lambda(K-S)^+ = 0, \qquad S > L^*, \tag{3.4}$$

together with the boundary conditions

$$\lim_{S\uparrow\infty} P^*(\lambda,S) = 0, \tag{3.5}$$
$$\lim_{S\downarrow L^*} P^*(\lambda,S) = K - L^*, \tag{3.6}$$
$$\lim_{S\downarrow L^*} P^*_S(\lambda,S) = -1. \tag{3.7}$$

The early exercise boundary $L^* \equiv L^*(\lambda)$ is given by the LCT of $\hat{B}_s = B_{T-s}$

$$L^*(\lambda) = \int_0^\infty \lambda e^{-\lambda s}\hat{B}_s ds, \tag{3.8}$$

which is a *constant* due to the *memoryless property* of the exponential distribution.

Theorem 1

$$P^*(\lambda,S) = \begin{cases} K - S, & S \le L^* \\ \dfrac{\lambda}{\lambda+r}K - \dfrac{\lambda}{\lambda+\delta}S + c(S) + b(S) + d(S), & L^* < S < K \\ p(S) + b(S) + d(S), & S \ge K, \end{cases} \tag{3.9}$$

where

$$c(S) = \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_-\right)K\left(\frac{S}{K}\right)^{\theta_+}, \tag{3.10}$$
$$p(S) = \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_+\right)K\left(\frac{S}{K}\right)^{\theta_-}, \tag{3.11}$$
$$b(S) = -\frac{\theta_+}{\theta_-}c(L^*)\left(\frac{S}{L^*}\right)^{\theta_-}, \tag{3.12}$$
$$d(S) = -\frac{1}{\theta_-}\frac{\delta}{\lambda+\delta}L^*\left(\frac{S}{L^*}\right)^{\theta_-}, \tag{3.13}$$

and the parameters $\theta_\pm$ are two roots of the quadratic equation $\frac{1}{2}\sigma^2\theta^2 + (r-\delta-\frac{1}{2}\sigma^2)\theta - (\lambda+r) = 0$, *i.e.*,

$$\theta_\pm = \frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2) \pm \sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r)}\right\}. \tag{3.14}$$

Proof. See Kimura [5]. □

Remark 2 The function $c(S)$ ($p(S)$) appeared in (3.9) can be interpreted as the randomized value of a European call (put) paying $(S-K)^+$ ($(K-S)^+$). Also, the function $b(S)$ ($d(S)$) can be interpreted as the present value of interest (dividends) received below the early exercise boundary L^*.

Remark 3 Carr's result for $b(S)$ (when $\delta = 0$) is *invalid.* The correct one is

$$b^{(1)}(S) = \left(\frac{S}{\underline{S}_1}\right)^{\gamma-\varepsilon} qK\left(RrT + \frac{1}{2\varepsilon p}\right)\left(\frac{\underline{S}_1}{K}\right)^{\gamma+\varepsilon}, \tag{3.15}$$

in terms of his notation; cf. [3, Equation (15)].

Theorem 2
(i) The early exercise boundary L^* of the Canadian-American put option satisfies the equation

$$\lambda\left(\frac{L^*}{K}\right)^{\theta_+} = r(\theta_+ - 1) - \delta\theta_+\frac{L^*}{K}. \tag{3.16}$$

(ii) For the limiting case $\lambda \to 0$, we have

$$L^*(0) = \lim_{s\to\infty}\hat{B}_s = \frac{r(\theta_+^\circ - 1)}{\delta\theta_+^\circ}K = \frac{\theta_-^\circ}{\theta_-^\circ - 1}K, \tag{3.17}$$

where $\theta_\pm^\circ = \lim_{\lambda\to 0}\theta_\pm$. In particular, if $\delta = 0$, then

$$L^*(0) = \lim_{s\to\infty}\hat{B}_s = \frac{K}{1+\dfrac{\sigma^2}{2r}}. \tag{3.18}$$

(iii) For the limiting case $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty} L^*(\lambda) = \hat{B}_0 = B_T = \min\left(\frac{r}{\delta}, 1\right)K. \tag{3.19}$$

Proof. See Kimura [5]. □

4 New Randomization Based on an Order Statistic

Let $X_1, \ldots, X_{n+m}$ be independent and exponentially distributed random variables with parameter α (> 0), and let $X_{(i)}$ denote the i-th smallest of these random variables ($i = 1, \ldots, n+m$). Then, the probability density function (pdf) of $X_{(n+1)}$ is

$$f(t) = \frac{(n+m)!}{n!(m-1)!}(1-e^{-\alpha t})^n \alpha e^{-m\alpha t}, \qquad t \geq 0. \tag{4.1}$$

The mean and variance of $X_{(n+1)}$ are given by

$$E[X_{(n+1)}] = \frac{1}{\alpha}\sum_{i=0}^{n}\frac{1}{m+i} \approx \frac{1}{\alpha}\ln\frac{2n+2m+1}{2m-1}, \tag{4.2}$$

$$V[X_{(n+1)}] = \frac{1}{\alpha^2}\sum_{i=0}^{n}\frac{1}{(m+i)^2} \approx \frac{1}{\alpha^2}\ln\frac{2n+2}{(2m-1)(2n+2m+1)}. \tag{4.3}$$

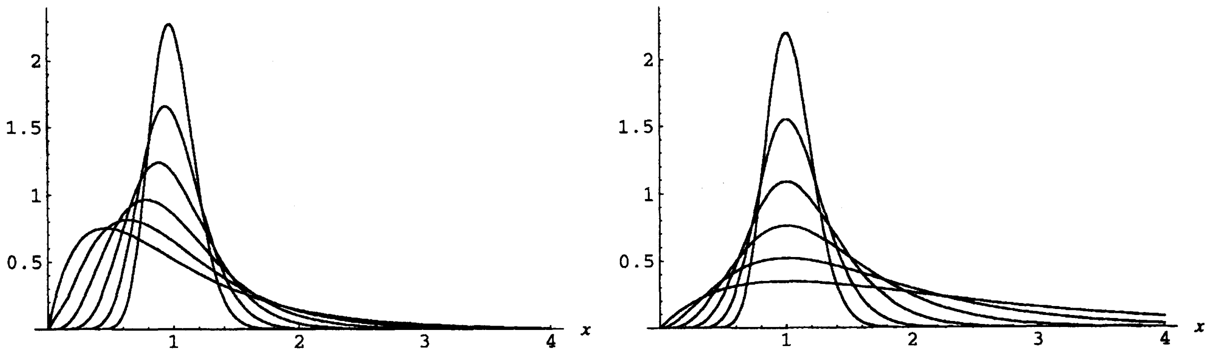
In addition, the modal value of $X_{(n+1)}$ is

$$M[X_{(n+1)}] \equiv \arg\max_t f(t) = \frac{1}{\alpha}\ln\frac{n+m}{m}. \tag{4.4}$$

If we let either $E[X_{(n+1)}] = T$ or $M[X_{(n+1)}] = T$, then $X_{(n+1)}$ can be another candidate for the random maturity $\widetilde{T}$, because $\lim_{n,m\to\infty} V[X_{(n+1)}] = 0$. For computational convenience, we adopt the mode matching $M[X_{(n+1)}] = T$, so that α can be determined as

$$\alpha = \frac{1}{T}\ln\frac{n+m}{m}. \tag{4.5}$$

Figure 2 shows the differences between the mean and mode matchings in the order-statistic-based randomization. From the figures (a) and (b), we find almost no differences between these matchings for large values of n.



(a) mean matching: $E[X_{(n+1)}] = 1$ (b) mode matching: $M[X_{(n+1)}] = 1$

Figure 2: The pdf of the order statistic $X_{(n+1)}$ ($n = m = 1, 2, 4, 8, 16, 32$)

For a continuous function $g(t)$ ($t \geq 0$), define

$$g^*_{n,m}(T) = \frac{(n+m)!}{n!(m-1)!}\int_0^\infty g(t)(1-e^{-\alpha t})^n \alpha e^{-m\alpha t}dt. \tag{4.6}$$

Then, we have

$$\lim_{n,m\to\infty} g^*_{n,m}(T) = g(T). \tag{4.7}$$

Theorem 3 The sequence $(g^*_{n,m})_{n,m\geq 1}$ satisfies the recursion

$$\begin{aligned} g^*_{0,m}(T) &= \int_0^\infty m\alpha e^{-m\alpha t} g(t)dt \\ g^*_{n,m}(T) &= \frac{n+m}{n}\, g^*_{n-1,m}(T) - \frac{m}{n}\, g^*_{n-1,m+1}(T), \qquad n \geq 1. \end{aligned} \tag{4.8}$$

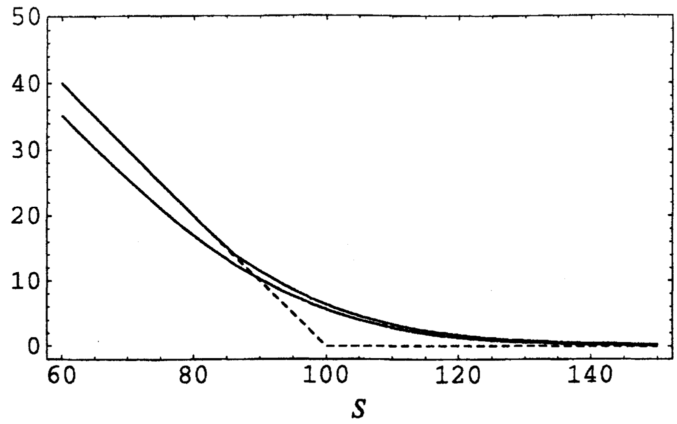


Figure 3: American & European put values
$(t = 0, K = 100, T = 1, r = 0.05, \delta = 0, \sigma = 0.2)$

Proof. See Kimura [5]. □

A simple and practical setting for the parameters n and m is $n = m$. For a set of the parameters $\{t, S, K, T, r, \delta, \sigma\}$, if we have a functional program for computing $P^*(\lambda, S)$ for any $\lambda \geq 0$, then the N-th randomized approximation $\pi_N \equiv g^*_{N,N} \approx P(t, S)$ $(N \geq 1)$ can be obtained by the following algorithm:

$\alpha = \frac{1}{T-t} \ln 2$
for $m = N$ **to** $2N$ **do**
  $g^*_{0,m} = P^*(m\alpha, S)$
next m
for $n = 1$ **to** N **do**
  for $m = N$ **to** $2N - n$ **do**
    $g^*_{n,m} = \frac{n+m}{n} g^*_{n-1,m} - \frac{m}{n} g^*_{n-1,m+1}$
  next m
next n
$\pi_N = g^*_{N,N}$

In order to speed up the convergence of N-th randomized approximation π_N, Carr [3] suggested using the Richardson extrapolation scheme. In this paper, however, we use another extrapolation scheme defined below, from the error analysis of the N-th approximation π_N; see Kimura [5] for details.

$$\begin{cases} \pi_N^{(0)} = \pi_N, & N = 2^0, 2^1, 2^2, \ldots \\ \pi_N^{(k)} = \dfrac{1}{2^k - 1}\left\{2^k \pi_N^{(k-1)} - \pi_{\frac{N}{2}}^{(k-1)}\right\}, & N = 2^k, 2^{k+1}, \ldots, \quad k \geq 1. \end{cases} \tag{4.9}$$

Figure 3 illustrates the curve of an American put and the associated European put values as a function of the present asset value S.

References

[1] Black, F., and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, **81** (1973) 637–654.

[2] Broadie, M., and J. Detemple, "American option valuation: new bounds, approximations, and a comparison of existing methods," *Review of Financial Studies*, **9** (1996) 1211–1250.

[3] Carr, P., "Randomization and the American put," *Review of Financial Studies*, **11** (1998) 597–626.

[4] Carr, P., and M. Chesney, "American put call symmetry," working paper, Morgan Stanley, 1997.

[5] Kimura, T., "Alternative randomization for valuing American options," working paper, Hokkaido University, 2004.

[6] McDonald, R., and M. Schroder, "A parity result for American options," *Journal of Computational Finance*, **1** (1998) 5–13.

[7] McKean, H.P., "Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics," *Industrial Management Review*, **6** (1965) 32–39.

[8] Merton, R., "The theory of rational option pricing," *Bell Journal of Economics and Management Science*, **4** (1973) 141–183.

[9] Widder, D.V., *The Laplace Transform*, Princeton University Press, Princeton, N.J., 1946.